

A note on convergence analysis of NURBS curve when weights approach infinity

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Abstract

This article considers the convergence of NURBS curve when weights approach infinity. We shows that limit of NURBS curve dose not exist when independent variables weights approach infinity. Further, pointwise convergence uniform convergence and L^1 convergence are researched.

Keywords: NURBS curve; Limit of Multivariable; Pointwise Convergence; L^1 Convergence;

1. Introduction

NURBS curve is the key tools in Computer Aided Geometric Design (CAGD). Let knot vector $\mathbf{U} = \{u_0, \dots, u_m\}$ be a non-decreasing sequence of real numbers. The p th-degree NURBS curve on closed interval $[u_i, u_{i+1}]$ can be defined as

$$\mathbf{C}(u) = \frac{\sum_{j=i-p}^i N_{j,p}(u)\omega_j \mathbf{p}_j}{\sum_{j=i-p}^i N_{j,p}(u)\omega_j} = \sum_{j=i-p}^i R_{j,p}^\omega(u) \mathbf{p}_j$$

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where $\{\mathbf{p}_j\}$ are control points, $\{\omega_j\}$ are weights, $\{N_{j,p}(u)\}$ are the p th-degree B-spline basis and $\{R_{j,p}^\omega(u)\}$ are rational basis functions.

The weights ω_j are typically used as shape parameters [1]. When ones of ω_j approach infinity, some results were obtain. For example, using weighted least squares, Pieg1 and Tiller[2] obtain that the NURBS curve approaches approximated data points and the limit curve consists of segment lines. Goldman [3] prove the NURBS curve will approaches the three control points. Here, we give another two results.

Example 1. Let $\mathbf{U} = \{0, 0, 0, 0, 1, 1, 1, 1\}$ and $\boldsymbol{\omega} = \{1, \omega_1, \omega_2, 1\}$. If $\omega_2 \asymp k\omega_1, k \neq 0$, we have the following result for cubic NURBS curve as $\omega_1 \rightarrow +\infty$.

$$\lim_{\substack{\omega_1 \rightarrow +\infty \\ \omega_2 \rightarrow +\infty}} \frac{\sum_{j=i-p}^i N_{j,p}(u) \omega_j \mathbf{p}_j}{\sum_{j=i-p}^i N_{j,p}(u) \omega_j} = \begin{cases} \mathbf{p}_0 & u = u_3 \\ \frac{N_1(u) \mathbf{p}_1 + k N_2(u) \mathbf{p}_2}{N_1(u) + N_2(u)} & u \in (u_3, u_4) ; \\ \mathbf{p}_3 & u = u_4 \end{cases}$$

if $\omega_2 \asymp k\omega_1^2$, we have the result for $\omega_1 \rightarrow +\infty$.

$$\lim_{\substack{\omega_1 \rightarrow +\infty \\ \omega_2 \rightarrow +\infty}} \frac{\sum_{j=i-p}^i N_{j,p}(u) \omega_j \mathbf{p}_j}{\sum_{j=i-p}^i N_{j,p}(u) \omega_j} = \begin{cases} \mathbf{p}_0 & u = u_3 \\ \mathbf{p}_2 & u \in (u_3, u_4) . \\ \mathbf{p}_3 & u = u_4 \end{cases}$$

In this paper, we using analysis methods to explain these results.

2. Main results

From the limit definition about multivariable and the **Example 1**, we can say that

Theorem 1. *The limit of NURBS curve does not exist when the j ($1 < j \leq p+1$) independent variables ω_j approach infinity.*

However, if the weights ω_j along an approach path we have the following theorems.

Theorem 2. *The NURBS curve $\mathbf{C}(u)$ on the interval $[u_i, u_{i+1}]$ is convergent pointwise when weights $\omega_j \rightarrow +\infty$.*

Proof. For a fixed point $u \in [u_i, u_{i+1})$, dividing the weights $\{\omega_i\}$ into two sets $\omega_{j_0} = \{\omega_{j_0}\}$ and $\omega_{j_1} = \{\omega_{j_1}\}$, where elements in ω_{j_0} have the same order of magnitude, that is, $\frac{\omega_{j_0}}{k_{j_0}} \asymp \min \omega_{j_0}$ ($j_0 \in \mathbf{j}_0$), and are negligible with respect to ones in ω_{j_1} , we have

$$\lim_{\omega_{j_0} \rightarrow \infty} \mathbf{C}(u) = \begin{cases} \frac{\sum_{j \in \mathbf{j}_0 - \{i+1\}} N_{j,p}(u) k_j \mathbf{p}_j}{\sum_{j \in \mathbf{j}_0 - \{i+1\}} N_{j,p}(u) k_j} & u = u_i \\ \frac{\sum_{j \in \mathbf{j}_0} N_{j,p}(u) k_j \mathbf{p}_j}{\sum_{j \in \mathbf{j}_0} N_{j,p}(u) k_j} & u \in (u_i, u_{i+1}) , \\ \frac{\sum_{j \in \mathbf{j}_0 - \{i\}} N_{j,p}(u) k_j \mathbf{p}_j}{\sum_{j \in \mathbf{j}_0 - \{i\}} N_{j,p}(u) k_j} & u = u_{i+1} \end{cases}$$

which is the desired conclusion. \square

Theorem 3. *If the knot vector $\tilde{\mathbf{U}} = \{u_0, \dots, u_m\}$ be a strictly increasing sequence, where $i - p \notin \mathbf{j}_0 \wedge i \notin \mathbf{j}_0$, the NURBS curve on the interval $[u_i, u_{i+1}]$ is uniformly convergent. the limit curve is as following*

$$\lim_{\omega_{j_0} \rightarrow \infty} \mathbf{C}(u) = \sum_{j \in \mathbf{j}_0} N_{j,p}(u) k_j \mathbf{p}_j \Big/ \sum_{j \in \mathbf{j}_0} N_{j,p}(u) k_j, \quad u \in [u_i, u_{i+1}].$$

Proof. Without loss of generality, consider one weight ω_k ($k \neq i - p \wedge k \neq i$) approaches positive infinity. When $\omega_k > \Omega(\varepsilon) = \frac{1}{\varepsilon} \frac{M}{m}$, where

$$M = \max \sum_{j=i-p \wedge j \neq k}^i N_{j,p}(u) \omega_j, \quad m = \min N_{k,p}(u),$$

we have

$$|R_{j,p}^\omega - 1| = \left| \frac{\sum_{j=i-p \wedge j \neq k}^i N_{j,p}(u)\omega_j}{\sum_{j=i-p}^i N_{j,p}(u)\omega_j} \right| < \left| \frac{\sum_{j=i-p \wedge j \neq k}^i N_{j,p}(u)\omega_j}{N_{k,p}(u)\omega_k} \right| < \varepsilon$$

or equivalently,

$$\omega_k > \frac{1}{\varepsilon} \frac{\sum_{j=i-p \wedge j \neq k}^i N_{j,p}(u)\omega_j}{N_{k,p}(u)} = \frac{1}{\varepsilon} \frac{M}{m},$$

where $N_{k,p}(u) \neq 0$ on the interval $[u_i, u_{i+1}]$ of knot vector $\tilde{\mathbf{U}}$. By the properties of continuous function on the closed interval, M and m do not depend on the parameter u , which imply the $R_{j,p}^\omega$ is uniformly convergent and then the NURBS curve is also uniformly convergent. \square

Finally, with the convex hull property of NURBS curve and Bounded convergence theorem [4], we obtain

Theorem 4. *The NURBS curve $\mathbf{C}(u)$ on the interval $[u_i, u_{i+1}]$ is L^1 convergence when $\omega_{j_0} \rightarrow +\infty$.*

Conclusion

We analysis convergence of NURBS curve in this paper. The geometric meaning of limit curve is our future research.

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